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GROUND STATE ESTIMATIONS IN GAUGE THEORY

BY

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ABSTRACT. – The lowest eigenvalue $\lambda(F)$ of a vector bundle F over a compact Riemannian manifold M is estimated in terms of the curvature of M and of the connection Γ on F. © 2001 Éditions scientifiques et médicales Elsevier SAS

Introduction

Given a compact Riemannian manifold M and an Euclidean vector bundle $F \to M$, we choose an Euclidean connection Γ over F. Then the Laplace–Beltrami operator Δ of M is lifted through the connection Γ into an elliptic operator $\widetilde{\Delta}$ operating on the sections of F. We shall discuss in this paper the problem of estimating the lowest eigenvalue $\lambda(F)$ in terms of the curvature of M and of Γ. The first part is devoted to the majoration of the ground state under an assumption of Ricci positivity of the base manifold M; we shall get such a majoration in the case of abelian gauge group; on the contrary a counter example using spin representation will show that such majoration does not remain true in the case of non abelian gauge group.

The second part will establish a universal minoration, which valids even if M is non-simply connected, and in non-abelian gauge theory. We shall develop this estimate at the level of frame bundle; but our methodology can be applied to the case where the gauge is an arbitrary principal bundle over M.

1. Magnetic field and Shigekawa identity

For a compact Riemannian manifold M and a differential 1-form A on M, the Schrödinger operator with vector potential A is a second order differential operator given by

$$H_A f = -\frac{1}{2}\Delta_M f - \sqrt{-1}\,\langle df, A\rangle + \left(\frac{\sqrt{-1}}{2}d^*A + \frac{1}{2}|A|^2\right)f,$$

where Δ_M is the Laplace–Beltrami operator on M, $\langle\cdot,\cdot\rangle$ is the inner product in the cotangent bundle T^*M inherited from the Riemannian metric, and $|\cdot|$ denotes the associated norm.

THEOREM 1.1. – *Let M be a compact Riemannian manifold, and H be a magnetic field, i.e., a closed differential 2-form having no harmonic part. Assume that the Ricci curvature* Ric *satisfies*

$$\mathrm{Ric} \geqslant c \times \mathrm{Identity} \quad \textit{for some } c > 0.$$

Then, for any differential 1-form A with $dA = H$, the ground state energy level μ of the Schrödinger operator with vector potential A enjoys that

$$\mu \leqslant \frac{1}{2c\,v_M(M)}\|H\|_{L^2}^2,$$

where v_M denotes the volume element of M, and $\|H\|_{L^2}$ does the L^2-norm of H with respect to v_M.

Proof. – Let $\delta = d^*$, the adjoint of the exterior derivative d, and define the de Rham–Hodge Laplacian Δ_k acting on differential k-forms by $\Delta_k = d\delta + \delta d$. Due to the Weitzenböck formula and the Ricci positivity, it holds that $\Delta_1 \geqslant c \times \mathrm{Identity}_{\mathrm{End}(L^2)}$. Hence Δ_1 is invertible on L^2 and there is no harmonic 1-form. Put

$$A_0 = \Delta_1^{-1}(\delta H).$$

Since H has no harmonic part, $dA_0 = H$. Moreover, $\delta A_0 = 0$, because Δ_1 preserves the range of δ and $\delta^2 = 0$. Furthermore, if A is a differential 1-form with $dA = H$, then there exists φ such that $A = A_0 + d\varphi$. Then, by virtue of [7, Theorem 4.3] and the gauge invariance, we have that $\mu \leqslant \mu'$, where μ' is the smallest eigenvalue of the operator $\frac{1}{2}\Delta_0 + \frac{1}{2}\|A_0(\cdot)\|^2$, $\|A_0(x)\|$ being the norm of $A(x)$ in T_x^*M. The eigenvalue μ' corresponds to minimizing the Dirichlet form

$$\int_M \left\{ \|d\phi(x)\|^2 + \frac{1}{2}\|A_0(x)\|^2|\phi(x)|^2 \right\} v_M(dx)$$

over $\phi \in C^\infty(M)$ with $\|\phi\|_{L^2} = 1$. Hence, substituting a test function $\phi \equiv 1/\sqrt{v_M(M)}$, we obtain

$$\mu' \leqslant \frac{\|A_0\|_{L^2}^2}{2v_M(M)}. \tag{1.1}$$

Due to the Weitzenböck formula and the Ricci positivity, we obtain that

$$\|H\|_{L^2}^2 = \|dA_0\|_{L^2}^2 = \int_M \langle A_0(x), \Delta_1 A_0(x)\rangle_{T_x^*M} v_M(dx) \geqslant c\|A_0\|_{L^2}^2.$$

Plugging this into (1.1) we obtain the desired estimation. □

Remark 1.1. – Lévy's formula implies that, on $\mathbb{R}^2$, the constant magnetic field $H = \lambda dx \wedge dy$ of intensity $\lambda \in \mathbb{R}$ has the ground state energy level $|\lambda|/2$. By Theorem 1.1, in the case of compact manifold with positive Ricci curvature, we get an asymptotic behaviour of order $O(\lambda^2)$ as $\lambda \to 0$.

The Aharonov–Bohm effect appears in a solenoidal circuit, which comes out of the Ricci positivity.

2. Flat spin bundle with a positive ground state

Let $\mathcal{Q}$ be the skew field of quaternion numbers. It is generated by e, i, j, and k, e being the unit element, and the algebraic operations are given by

$$\begin{aligned}
&(te + xi + yj + zk) + (t'e + x'i + y'j + z'k) \\
&\quad = (t + t')e + (x + x')i + (y + y')j + (z + z')k, \\
&ij = k, \quad jk = i, \quad ki = j, \quad i^2 = j^2 = k^2 = -e.
\end{aligned}$$

A quaternion number of the form $xi + yj + zk$ is called a pure quaternion number, and the set of all pure quaternion numbers is denoted by $\mathcal{Q}_0$. For $q = te + xi + yj + zk \in \mathcal{Q}$, its conjugate $\bar{q}$ is defined by $\bar{q} = te - xi - yj - zk$, and its norm is given by $|q| = \sqrt{q\bar{q}}$. Set $\mathcal{Q}_1 = \{q \in \mathcal{Q}\colon |q| = 1\}$. Since $|qq'| = |q||q'|$, $\mathcal{Q}_1$ is a multiplicative group having a center $Z = \{e, -e\}$. Identifying $\mathcal{Q}_0$ with $\mathbb{R}^3$, we define a homeomorphism $\Psi : \mathcal{Q}_1 \to \mathrm{SO}(3)$ by

$$\Psi(\sigma)[q] = \sigma q \sigma^{-1}, \quad \sigma \in \mathcal{Q}_1,\ q \in \mathcal{Q}_0.$$

Then $\ker(\Psi) = Z$ and $\mathcal{Q}_1/Z \simeq \mathrm{SO}(3)$.

The spin representation is the group homeomorphism $\theta : \mathcal{Q}_1 \to \mathrm{SO}(4)$ defined by $\theta(\sigma)[q] = \sigma q$, where $\sigma \in \mathcal{Q}_1$, $q \in \mathcal{Q}$, and we have identified $\mathcal{Q}$ with $\mathbb{R}^4$. Denoting by $\Omega = g^{-1}dg$ the left invariant Maurer–Cartan differential form on SO(4), we define an $\mathfrak{so}(4)$-valued differential 1-form Γ on $\mathcal{Q}_1$ by

$$\Gamma = \theta^* \Omega.$$

Since θ^* commutes with d and Lie brackets, the form Γ satisfies the Darboux structural equation

$$d\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0.$$

LEMMA 2.1. – *There exists an* $\mathfrak{so}(4)$*-valued differential* 1*-form* Γ_1 *on* SO(3) *such that* $\Psi^*\Gamma_1 = \Gamma$*. Furthermore,* Γ_1 *satisfied the Darboux structural equation;* $d\Gamma_1 + \frac{1}{2}[\Gamma_1, \Gamma_1] = 0$.

Proof. – For $\alpha \in \mathrm{SO}(3)$, $\Psi^{-1}(\{\alpha\}) = \{\sigma_0, \sigma_1\}$ for some $\sigma_0, \sigma_1 \in \mathcal{Q}_1$ with $\sigma_1 = -\sigma_0$. Given $u \in \mathfrak{so}(3)$, we take $v_i \in \mathcal{Q}_0$ so that $\Psi(\exp(\varepsilon v_i)\sigma_i) = \exp(\varepsilon u)\alpha$, $i = 0, 1$. Differentiating this relation at $\varepsilon = 0$, we obtain

$$v_i \sigma_i q \sigma_i^{-1} - \sigma_i q \sigma_i^{-1} = u\alpha(q).$$

Since the left hand side does not change when we replace σ_0 by $-\sigma_0$, the assertion follows. □

Note that $\Psi : \mathcal{Q}_1 \to \mathrm{SO}(3)$ is an O(1)-bundle. Identifying (q, y) and $(-q, -y)$ in $\mathcal{Q}_1 \times \mathbb{R}^4$, we then obtain a vector bundle F over SO(3) with fiber $\mathbb{R}^4$. Let $\{O_k\}$ be a finite covering of $\mathcal{Q}_1$ such that each restriction Ψ_k

of Ψ to O_k is injective. Put $\Omega_k = \Psi(O_k)$ and consider the trivial vector bundle $F_k = O_k \times \mathbb{R}^4$ over O_k. Then, via the reciprocal image $(\Psi_k^{-1})^* F_k$, we get a coherent family of vector bundles over Ω_k, which determines the vector bundle F over SO(3) as above.

Let $E = \mathcal{Q}_1 \times \mathbb{R}^4$ be the trivial bundle over $\mathcal{Q}_1$. The space of smooth sections $\Gamma^\infty(E)$ of the vector bundle E over $\mathcal{Q}_1$ can be identified with $C^\infty(\mathcal{Q}_1; \mathbb{R}^4)$. Under this identification, the covariant derivative $\nabla^{(1)}$ is defined by $\nabla^{(1)} s = ds + \Gamma \cdot s$, $s \in C^\infty(\mathcal{Q}_1; \mathbb{R}^4)$. By the definition of the vector bundle F over SO(3), denoting by $\Gamma^\infty(F)$ the smooth sections of F, we have the identification

$$\Gamma^\infty(F) = \left\{ s \in C^\infty\left(\mathcal{Q}_1; \mathbb{R}^4\right):\ s(-q) = -s(q),\ q \in \mathcal{Q}_1 \right\}. \tag{2.1}$$

By Lemma 2.1, we see that the covariant derivative $\nabla^{(1)}$ admits a restriction to $\Gamma^\infty(F)$, say ∇^F, and that the curvature associated with ∇^F vanishes.

Under the identification between $\mathcal{Q}$ and $\mathbb{R}^4$, $\mathcal{Q}_1$ is identified with the 3-sphere S^3. Hence it is equipped with the Riemannian metric inherited from the standard flat metric of $\mathbb{R}^4$. Then $\mathcal{Q}_1$ is a compact manifold with positive Ricci curvature. The Riemannian metric on $\mathcal{Q}_1$ induces that on SO(3). Consider the Brownian loop $m_\omega(t)$ on SO(3) starting and ending at the identity at time 0 and T, respectively. It is lifted to the process q_ω on $\mathcal{Q}_1$ with $q_\omega(0) = e$ and $q_\omega(T) \in Z = \{\pm e\}$. Let ν_T be the law of $q_\omega(T)$ on Z.

LEMMA 2.2. – *There exists $C, c > 0$, which is independent of T, such that*

$$0 < \nu_T(\{e\}) - \nu_T(\{-e\}) \leqslant C \exp(-cT). \tag{2.2}$$

Proof. – Let $p(t, q, q')$ be the heat kernel associated with the Laplace–Beltrami operator on $\mathcal{Q}_1$. Then we have

$$\begin{aligned} \nu_T(\{e\}) &= \frac{p(T, e, e)}{p(T, e, e) + p(T, e, -e)}, \\ \nu_T(\{-e\}) &= \frac{p(T, e, -e)}{p(T, e, e) + p(T, e, -e)}. \end{aligned} \tag{2.3}$$

Due to the Ricci positivity of $\mathcal{Q}_1$, there exists $C, c > 0$ such that

$$\sup_{q\in\mathcal{Q}_1}\left|\int_{\mathcal{Q}_1} f(r)p(T,q,v)v_{\mathcal{Q}_1}(dv) - \frac{1}{v_{\mathcal{Q}_1}(\mathcal{Q}_1)}\int_{\mathcal{Q}_1} f(v)v_{\mathcal{Q}_1}(dv)\right| \leqslant C\|df\|_\infty \mathrm{e}^{-cT} \tag{2.4}$$

for any $T>0$ and $f\in C^\infty(\mathcal{Q}_1)$, where $v_{\mathcal{Q}_1}$ stands for the volume element of $\mathcal{Q}_1$ and $\|\cdot\|_\infty$ does the uniform norm on $\mathcal{Q}_1$. For example, see [8, Theorem 6.22]. Substituting $p(1,\cdot,e)-p(1,\cdot,-e)$ and $p(1,\cdot,e)$ for f in (2.4), we obtain that

$$\left|p(T+1,e,e)-p(T+1,e,-e)\right| \leqslant C\left\|d\left[p(1,\cdot,e)-p(1,\cdot,-e)\right]\right\|_\infty \mathrm{e}^{-cT}$$

and

$$\sup_{q\in\mathcal{Q}_1}\left|p(T+1,e,q)-\frac{1}{v_{\mathcal{Q}_1}(\mathcal{Q}_1)}\right| \leqslant C\left\|d[p(1,\cdot,e)]\right\|_\infty \mathrm{e}^{-cT} \tag{2.5}$$

for any $T>0$. In conjunction with (2.3), these imply the upper estimation in (2.2).

To see the lower estimation in (2.2), let $q'_\omega(t)$ be the Brownian motion on $\mathcal{Q}_1$ starting at e and $m_\omega(t)=\Psi(q'_\omega(t))$. Denoting by $\mathbb{P}$ the underlying probability measure, we obtain

$$\nu_T(\{e\}) = \mathbb{P}\left(q'(T)=e \mid m'(T)=\mathrm{Id}\right),$$
$$\nu_T(\{-e\}) = \mathbb{P}\left(q'(T)=-e \mid m'(T)=\mathrm{Id}\right).$$

Set $\Lambda=\inf\{t\geqslant 0\colon\ q'(t)\in\mathcal{Q}_0\cap\mathcal{Q}_1\}$. By virtue of the Markov property and the symmetry of the Brownian motion on $\mathcal{Q}_1$, we have

$$\mathbb{P}\left(q'(T)=e,\Lambda<T \mid m'(T)=e\right)=\mathbb{P}\left(q'(T)=-e,\Lambda<T \mid m'(T)=e\right).$$

Since $\mathbb{P}(q'(T)=-e,\Lambda\geqslant T \mid m'(T)=\mathrm{Id})=0$, we arrive at

$$\nu_T(\{e\})-\nu_T(\{-e\}) = \mathbb{P}\left(q'(T)=e,\Lambda\geqslant T \mid m'(T)=\mathrm{Id}\right)>0.$$

Thus the lower estimation in (2.2) has been verified. □

THEOREM 2.1. – *The bundle F with the covariant derivative ∇^F as described above has a null curvature, and the base manifold* SO(3) *has a strictly positive Ricci curvature. Moreover, the ground state $\lambda(F)$ of the covariant Laplacian associated with ∇^F satisfies $\lambda(F)\geqslant c$, where $c>0$ is the constant which appeared in Lemma* 2.2.

Proof. – Let $k(t,m,m')$ be the heat kernel for the covariant Laplacian associated with ∇^F. As in the previous lemma, we denote by $q'_\omega(t)$ the Brownian motion on $\mathcal{Q}_1$ and set $q'_\omega(t,q) = q'_\omega(t)q$ for $q \in \mathcal{Q}_1$. Define $e_\omega(t,q) \in \mathrm{SO}(4)$ by

$$de_\omega(t,q) = -\Gamma\big(q'_\omega(t,q)\big)e_\omega(t,q) \circ dq'_\omega(t,q), \quad e_\omega(0,q) = \mathrm{Id}.$$

Then, by the identification (2.1), for q, q' with $\Psi(q) = m$ and $\Psi(q') = m'$, we have

$$k(t,m,m') = \mathbb{E}\big[e_\omega(t,q)^{-1}\big\{\delta_{q'}\big(q'_\omega(t,q)\big) - \delta_{-q'}\big(q'_\omega(t,q)\big)\big\}\big],$$

where $\delta_q(\cdots)$ stands for Watanabe's pullback of the Dirac function via $q'_\omega(t,q)$. This yield that

$$k(t,m,m) = \mathbb{E}\big[e_\omega(t,q)^{-1}\big\{\delta_e\big(q'_\omega(t)\big) - \delta_{-e}\big(q'_\omega(t)\big)\big\}\big].$$

Thus we obtain

$$\big\langle \xi, k(t,m,m)\xi \big\rangle = |\xi|^2\big\{p(t,e,e) + p(t,e,-e)\big\}\big\{\nu_T(\{e\}) - \nu_T(\{-e\})\big\}$$

for any $\xi \in \mathbb{R}^4$, which, in conjunction with Lemma 2.2 and (2.5), implies the desired estimation of $\lambda(F)$. □

3. Anticipative tangent process and transversal analytic hypoellipticity

3.1. Calculus of variation on $O(M)$

Let X be the Wiener space of the d-dimensional Brownian motion and μ be the Wiener measure on it. An anticipating tangential process is a process ζ on X of the form

$$d\zeta^\alpha(\tau) = a^\alpha_\beta(\tau)\,dx^\beta(\tau) + c^\alpha(\tau)\,d\tau,$$

where $a^\alpha_\beta = -a^\beta_\alpha$, $\alpha^\alpha_\beta(0) = 0$, $c^\alpha(0) = 0$, $\mathbb{E}[\int_0^1 |c|^2\,d\tau] < \infty$, and a^α_β may not be adapted, and hence the integral should be understood as a Skorohod stochastic integral (the standard summation convention has been used).

Let $O(M)$ be the the orthonormal frame bundle over M. Then $O(M)$ admits the canonical parallelization; i.e., there exists a differential 1-form $\Theta = (\dot{\Theta}, \ddot{\Theta})$ with values in $\mathbb{R}^d \times \mathfrak{so}(d)$ such that, for every $r \in O(M)$, Θ_r is an isomorphism of $T_r O(M)$ onto $\mathbb{R}^d \times \mathfrak{so}(d)$. Using this parallelism, we can define, the horizontal Brownian motion $r_x(\tau, r_0)$ on $O(M)$ starting at r_0, which defines a horizontal flow $U^x_{\tau \leftarrow 0}(r_0) := r_x(\tau, r_0)$.

Denoting by R and Ω the Ricci curvature and the curvature tensor of the Riemannian manifold M, we link two tangent processes ζ and ζ' by the relation

$$d\zeta = d\zeta' + \frac{1}{2} R(\zeta')\, d\tau + \rho\, dx(\tau), \tag{3.1.1}$$

$$d\rho = \Omega(\circ dx, \zeta'), \tag{3.1.2}$$

where R and Ω are both scalarized with the frame $r \in O(M)$, and evaluated at $r(\tau, r_0)$.

To handle the stochastic calculus of variation for anticipating tangent processes, we generalize the result for adapted tangent processes obtained in [1].

PROPOSITION 3.1.1. – *Let ζ, ζ' be as above. Then, an infinitesimal variation $x \mapsto x + \varepsilon\zeta$ of $x \in X$ induces a variation of the horizontal flow U^x, which is represented in terms of the canonical parallelism of $O(M)$ as follows*:

$$\big(\zeta'(*), \rho(*)\big). \tag{3.1.3}$$

Proof. – It is sufficient to treat the case $d\xi'_\alpha = \phi a_{\alpha,\beta}\, dx^\beta$ with a adapted and ϕ a smooth functional, the stochastic integral being a Skorohod integral. We denote $d\tilde{\xi}'_\alpha = \phi a_{\alpha,\beta}\, dx^\beta$, then

$$\xi'_\alpha(\tau) = \phi \int_0^\tau a_{\alpha,\beta}\, dx^\beta - h'_\alpha(\tau)$$

with $h'_\alpha(\tau) = \int_0^\tau (D_{\eta,\beta}\phi) a_{\alpha,\beta}\, d\eta$. For a smooth functional F on the path space,

$$(D_{\xi'} F) \circ I = (\phi \circ I) D_{\tilde{\xi}}(F \circ I) - D_h(F \circ I),$$

where

$$d\tilde{\xi} = d\tilde{\xi}' - \rho \circ dx, \quad d\rho = \Omega(\tilde{\xi}', \circ dx)$$

and

$$dh = dh' - \bar{\rho} \circ dx, \quad d\bar{\rho} = \Omega(h', \circ dx).$$

We have $(\phi \circ I)\tilde{\xi}_\alpha(\tau) = \xi'_\alpha(\tau) + h'_\alpha(\tau) - \zeta_\alpha(\tau)$, where

$$\zeta_\alpha(\tau) = \int_0^\tau \left[\int_0^\eta \Omega^{\gamma,\alpha}_{\lambda,\delta} \left(\phi \tilde{\xi}'\right)^\lambda \circ dx^\delta \right] \circ dx^\gamma$$

and

$$(D_{\xi'} F) \circ I = (D_{\tilde{\xi}} + D_{\int_0^\cdot \bar{\rho} \circ dx} - D_\zeta)(F \circ I) = D_\xi(F \circ I),$$

$$d\xi = d\tilde{\xi} - \rho^* \circ dx, \quad d\rho^*_\alpha = -\bar{\rho}_\alpha \circ dx + \left(\int_0^\tau \Omega^{\cdot,\alpha}_{\lambda,\delta} \left(\phi \tilde{\xi}'\right)^\lambda \circ dx^\delta \right) \circ dx.$$

Since $d\rho^*_\alpha = \int_0^\tau (\Omega^{\cdot,\alpha}_{\lambda,\delta}(\phi\xi' - h')^\lambda \circ dx^\delta) = \int_0^\tau \Omega^\alpha(\xi', \circ dx)$, the result is proved. □

COROLLARY 3.1.1. – *Define*

$$\Phi^\beta(x, \tau) = \int_0^\tau \Omega_{\alpha,\beta} \circ dx^\alpha,$$

where Ω is again the scalarization of the curvature tensor at $r_x(\tau, r_0)$, which is thought of as a mapping of $\mathbb{R}^d \times \mathbb{R}^d \to \mathfrak{so}(d)$, and $\Omega_{\alpha,\beta}$ is the evaluation of this map at (e_α, e_β), $\{e_\alpha\}$ being the standard coordinate of $\mathbb{R}^d$. If $\zeta'(t) = 0$, then

$$\rho(t) = -\int_0^t \Phi^\beta(\tau)\, d\zeta'_\beta(\tau).$$

Proof. – The assertion follows by applying an integration by parts in the variable τ. □

3.2. A variance analysis

Denote by $\{\varepsilon_{\gamma,\delta}; 1 \leqslant \gamma < \delta \leqslant d\}$ the canonical basis of $\mathfrak{so}(d)$, and simply we write θ for (γ, δ). The functional Φ^β defined in Corollary 3.1.1

takes values in $\mathfrak{so}(d)$ and is expressed as $\Phi^\beta = \sum_\theta \Phi_\theta^\beta \varepsilon_\theta$. Fixing $t \in (0,1)$, for a function Ψ on $[0,1]$, we set $\overline{\Psi} = \frac{1}{t}\int_0^t \Psi$ and $\widetilde{\Psi} = \Psi - \overline{\Psi}$. Throughout the remainder of this section, we do not indicate the dependence on t explicitly to keep the notation simple, while most quantities appearing later depend on t. Define a *covariance matrix* by

$$\sigma_\theta^{\theta'} = \sum_\beta \int_0^t \widetilde{\Phi}_\theta^\beta(\tau)\widetilde{\Phi}_{\theta'}^\beta(\tau)\,d\tau. \tag{3.2.1}$$

PROPOSITION 3.2.1. – *Let $\mathcal{A} \in \mathfrak{so}(d)$. If σ defined in* (3.2.1) *is invertible, then $\{c_\beta\}$ defined by*

$$c_\beta(\tau) = \sum_\theta a_\theta \widetilde{\Phi}_\theta^\beta(\tau), \quad \textit{where } a_\theta = \sum_{\theta'} \left[\sigma^{-1}\right]_\theta^{\theta'} \mathcal{A}_{\theta'}, \tag{3.2.2}$$

satisfy

$$\sum_\beta \int_0^t \widetilde{\Phi}_\theta^\beta(\tau) c_\beta(\tau)\,d\tau = \mathcal{A}_\theta. \tag{3.2.3}$$

Proof. – Substituting (3.2.2) into the left hand side of (3.2.3), we obtain

$$\sum_\theta \sigma_{\theta'}^\theta a_\theta = \mathcal{A}_{\theta'},$$

which implies the assertion. □

Using the above c_β's, we define a tangent process ζ' by $\zeta'_\beta(\tau) = \int_0^\tau c_\beta(s)\,ds$. Then $\zeta'(t) = 0$, and, by Corollary 3.1.1, $\rho(t) = \mathcal{A}$. By Lemma 3.1.1, the corresponding infinitesimal variation is due to ζ defined by (3.1.1) with these ζ' and ρ.

The process Φ^β itself is adapted, and the non-adaptedness comes from $\overline{\Phi}^\beta$ and σ^{-1}.

3.3. Stochastic calculus of variation for small time

We would like to estimate the behavior of σ^{-1} as $t \to 0$. We first notice that

$$\mathbb{E}\left[\sigma_\theta^{\theta'}\right] = \frac{t^2}{2}\Gamma_\theta^{\theta'}(r_0) + \mathrm{o}\left(t^2\right) \quad (t \to 0), \tag{3.3.1}$$

where $\Gamma_{\theta}^{\theta'}(r_0) = \sum_{\alpha,\beta} \Omega_{\alpha,\beta;\theta}(r_0)\Omega_{\alpha,\beta;\theta'}(r_0)$, but the identity (3.3.1) gives no a.s. estimation of σ^{-1}. To obtain an estimation of σ^{-1}, we need an assumption on the curvature tensor. To state this, note that, at every point $r \in O(M)$, the curvature tensor Ω determines a symmetric endomorphism of $\Lambda^2(\mathbb{R}^d)$, the space of differential 2-forms on $\mathbb{R}^d$. To emphasize that this endomorphism is considered, we write (Ω) or $(\Omega)_r$ with parentheses. Under the invertibility assumption of (Ω), we obtain the estimation of σ^{-1}.

LEMMA 3.3.1. – *Suppose that $(\Omega)_r$ is invertible for every $r \in O(M)$. Then, for every $p > 1$, there exist constants $C_p(\Omega) > 0$ such that*

$$\mathbb{E}\big[\|\sigma^{-1}\|^p\big] \leqslant C_p(\Omega)t^{-2p} \quad \textit{for any } t \in (0,1]. \tag{3.3.2}$$

Proof. – Let $r^t(\tau, x)$ be the horizontal Brownian motion on $O(M)$ time changed by t; $dr^t = \sqrt{t}L_\alpha(r^t) \circ dx^\alpha$, where $\{L_\alpha\}$ is the system of the canonical horizontal vector fields. Set

$$\Psi_\theta^\beta(\tau) = \int_0^\tau \Omega_{\alpha,\beta;\theta}\big(r^t(s)\big) \circ dx^\alpha(s).$$

Then, by the time change argument, we obtain

$$\sigma_\theta^{\theta'} = t^2 \sum_\beta \int_0^1 \widetilde{\Psi}_\theta^\beta(\tau)\widetilde{\Psi}_{\theta'}^\beta(\tau)\,d\tau.$$

Put

$$\widehat{\sigma}_\theta^{\theta'} = \sum_\beta \int_0^1 \widetilde{\Psi}_\theta^\beta(\tau)\widetilde{\Psi}_{\theta'}^\beta(\tau)\,d\tau.$$

Since (Ω) is invertible, we have

$$\inf_{r \in O(M), A \in \Lambda^2(\mathbb{R}^d); \|A\|=1} \big\|(\Omega)_r[A]\big\| > 0.$$

Hence there are a positive number $\varepsilon > 0$, an open covering $\{U_i\}_{i=1}^N$ of $O(M)$, and indices $1 \leqslant \beta_1, \ldots, \beta_N \leqslant d$ such that

$$\sum_\alpha \left| \sum_\theta \Omega_{\alpha\beta_i;\theta}(r) A_\theta \right|^2 \geqslant \varepsilon$$

for any $r \in U_i$, $A \in \Lambda^2(\mathbb{R}^d)$ with $\|A\| = 1$, and $i = 1, \ldots, N$. Then, in repetition of the standard argument to show the nondegeneracy of the Malliavin covariance for the solution to an SDE with elliptic diffusion coefficients (cf. [9]), we obtain

$$\sup_{t \in (0,1]} \mathbb{E}\left[\left\|\widehat{\sigma}^{-1}\right\|^p\right] < \infty \quad \text{for any } p > 1.$$

Thus the assertion has been shown. □

3.4. Approximate stochastic calculus of variations

We denote by Ω^0 the curvature tensor frozen at the point r_0, and approximate $\Phi^\beta(x,\tau)$ by

$$\sum_\alpha \Omega^0_{\alpha,\beta} x^\alpha(\tau).$$

Introduce a Gaussian process

$$\tilde{x}_\alpha(\tau) = x_\alpha(\tau) - 1_{[0,t)}(\tau) \int_0^t x_\alpha(s) \frac{ds}{t}. \tag{3.4.1}$$

Then $\widetilde{\Phi}^\beta$ is approximated by $\sum_\alpha \Omega^0_{\alpha,\beta} \tilde{x}_\alpha(\tau)$, and the covariance matrix σ is done by

$$\sigma_0 = \sum_{\alpha,\alpha',\beta} \Omega^0_{\alpha,\beta} \Omega^0_{\alpha',\beta} \gamma_{\alpha,\alpha'}(t), \tag{3.4.2}$$

where

$$\gamma_{\alpha,\alpha'}(t) = \int_0^t \tilde{x}_\alpha(\tau) \tilde{x}_{\alpha'}(\tau)\, d\tau.$$

LEMMA 3.4.1. – *The symmetric matrix $t^{-2}\gamma(t)$ is a.s. positive definite and has a law which is equal to the law of $\gamma(1)$. Furthermore, there exists $c > 0$, which can be explicitly computed, such that*

$$\mu\big(\big\|\gamma(t)^{-1}\big\| > Mt^{-2}\big) \leqslant \exp(-cM).$$

Proof. – The assertion can be found in [5] and [4]. □

The c_β in (3.2.2) are now approximated by

$$c_\beta^0 = \sum_{\alpha,\theta} a_\alpha^0 \Omega^0_{\alpha,\beta;\theta} \tilde{x}_\alpha, \quad \text{where } a_\theta^0 = \sum_{\theta'} \big[\sigma_0^{-1}\big]_\theta^{\theta'} \mathcal{A}_{\theta'}.$$

3.5. Estimating the divergence of a variation on the Wiener space

Using the above approximation method, we can estimate the divergence $\delta(\zeta)$ of ζ given in (3.1.1) with ζ' and ρ as described at the end of the Section 3.2. Setting $\lambda(t) = \sqrt{\gamma(t)}$, we have

$$\sigma_0 = \big\{(\Omega)_{r_0}(\lambda \otimes 1)\big\}\big\{(\Omega)_{r_0}(\lambda \otimes 1)\big\}^*. \tag{3.5.1}$$

THEOREM 3.5.1. – *The divergence $\delta(\zeta)$ satisfies*

$$\lim_{t\to 0} t\mathbb{E}\big[|\delta(\zeta)|\big] \leqslant c_d \big\|(\Omega)^{-1}\big\| \|\mathcal{A}\|, \tag{3.5.2}$$

where c_d is a universal positive constant depending only on the dimension d, and can be computed.

For the proof, we prepare a lemma;

LEMMA 3.5.1. – *Let x^t denote the Brownian motion on $[0,t]$ and $(u(x^t))(\eta) = \frac{1}{\sqrt{t}} x^t(t\eta)$ the isometry between the corresponding Wiener spaces. If ϕ is a differentiable functional of x^1 then $F^t(x^t) = \phi(x)$ is differentiable and the following rescaling property holds*:

$$D_\tau F^t\big(x^t\big) = \frac{1}{\sqrt{t}} D_{\frac{\tau}{t}} \phi\big(x^1\big).$$

In particular, it holds that

$$t\,\mathbb{E}\big(D_{Z_0^t} F^t\big(x^t\big)\big) = \mathbb{E}\big(D_{Z_0^1} \phi\big(x^1\big)\big).$$

Proof. – We shall prove the lemma for cylindrical smooth functionals ϕ, the result being obtained by closure.

If $\phi(x^1) = \phi(x^1(\tau_1), \ldots, x^1(\tau_m))$,

$$\begin{aligned} D^t_\tau F^t(x^t) &= \sum_k 1_{\tau < t\tau_k} \partial_k \phi\left(\frac{x^t(t\tau_1)}{\sqrt{t}}, \ldots, \frac{x^t(t\tau_m)}{\sqrt{t}}\right) \\ &= \sum_k 1_{\frac{\tau}{t} < \tau_k} \frac{1}{\sqrt{t}} \partial\phi\big(x^1(\tau_1), \ldots, x^1(\tau_m)\big) \\ &= D_{\frac{\tau}{t}} \frac{1}{\sqrt{t}} \phi\big(x^1\big). \end{aligned}$$

Since the matrix $\sigma^t_0 = \Omega_{\alpha,\beta}\Omega_{\alpha',\beta} \int_0^t \tilde{x}^t_\alpha(\eta)\tilde{x}^t_{\alpha'}(\eta)\, d\eta$ rescales as $\sigma^t_0 = t^2\sigma^1_0$, we have

$$\begin{aligned} &\mathbb{E}\big(D_{Z^t_0} F^t(x^t)\big) \\ &\quad = \mathbb{E}\left[\int_0^t D^t_\tau F^t(x^t)\big[\sigma^t_0\big]^{-1} \tilde{x}^t(\tau)\, d\tau\right] \\ &\quad = \mathbb{E}\left[\int_0^t \frac{1}{\sqrt{t}} D_{\frac{\tau}{t}} \phi t^{-2}\big[\sigma^1_0\big]^{-1}\left(\sqrt{t} x^1\left(\frac{\tau}{t}\right) - \frac{1}{t}\int_0^t \sqrt{t} x\left(\frac{s}{t}\right) ds\right) d\tau\right] \\ &\quad = \mathbb{E}\left[\int_0^1 D_\lambda \phi t^{-2}\big[\sigma^1_0\big]^{-1} \tilde{x}^1(\lambda)\, t\, d\lambda\right]. \qquad \square \end{aligned}$$

Proof of Theorem 3.5.1. – We shall estimate the L^2 norms of the divergence of the vector fields by using the equality

$$\mathbb{E}\big[|\delta Z|^2\big] = \mathbb{E}\left[\int |\dot{Z}|^2\right] + \mathbb{E}\left[\int\!\!\int D_\tau \dot{Z}(\eta).D_\eta \dot{Z}(\tau)\right].$$

Working with the curvature tensor frozen at r_0 and at time $t = 1$ we have the following estimate for the L^2 norm of the corresponding approximating vector field ζ_0:

$$\mathbb{E}\left[\int_0^1 \sum_\beta \left|\big[\sigma_0^{-1}\big]_\theta^{\theta'} \mathcal{A}_{\theta'} \widetilde{\Phi}^\beta_\theta(\eta)\right|^2 d\eta\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^1 \sum_\beta \left[\sigma_0^{-1}\right]_\theta^{\theta'} \mathcal{A}_{\theta'} \widetilde{\Phi}_\theta^\beta(\eta)\left[\sigma_0^{-1}\right]_{\theta_1}^{\theta_1'} \mathcal{A}_{\theta_1'} \widetilde{\Phi}_{\theta_1}^\beta(\eta)\, d\eta\right] \\
&= \mathbb{E}\left(\left[\sigma_0^{-1}\right]_\theta^{\theta'} \mathcal{A}_{\theta'}\left[\sigma_0^{-1}\right]_{\theta_1}^{\theta_1'} [\sigma_0]_\theta^{\theta_1} \mathcal{A}_{\theta_1'}\right) \\
&= \mathbb{E}\left(\left\langle\left[\sigma_0^{-1}\right] \mathcal{A}, \mathcal{A}\right\rangle\right) \leqslant E\left\|\sigma_0^{-1}\right\|\|\mathcal{A}\|^2.
\end{aligned}
$$

We now compute the derivatives of σ and of $\widetilde{\Phi}$; we have

$$
\begin{aligned}
D_{\tau,\alpha} \widetilde{\Phi}_\theta^\beta(\eta) &= D_{\tau,\alpha} \sum_{\alpha'} \Omega_{\alpha',\beta}^\theta \left(x_{\alpha'}(\eta) - \int_0^1 x_{\alpha'}\right) \\
&= \Omega_{\alpha,\beta}{}^\theta \left(1_{\tau<\eta} - (1-\tau)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
[D_{\tau,\alpha}\sigma_0]_\theta^{\theta'} &= \left[D_{\tau,\alpha}\left(\sum_{\gamma,\gamma',\delta} \Omega_{\gamma,\delta}^\theta \Omega_{\gamma',\delta}^{\theta'} \int_0^1 \tilde{x}_\gamma(\eta)\tilde{x}_{\gamma'}(\eta)\, d\eta\right)\right] \\
&= \sum_{\gamma,\delta}\left(\Omega_{\gamma,\delta}^\theta \Omega_{\alpha,\delta}^{\theta'} + \Omega_{\alpha,\delta}^\theta \Omega_{\gamma,\delta}^{\theta'}\right) \int_0^1 \tilde{x}_\gamma(\eta)\left(1_{\tau<\eta} - (1-\tau)\right) d\eta.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^1 \int_0^1 \sum_{\alpha,\beta} |D_{\tau,\alpha} c_\beta(\eta)|^2\, d\tau\, d\eta\right] \\
&\quad \leqslant 2\mathbb{E}\left[\int_0^1 \int_0^1 \sum_{\alpha,\beta} \left|\left[\sigma_0^{-1}\right]_\theta^{\theta'} \mathcal{A}_{\theta'} \Omega_{\alpha,\beta}^\theta \left(1_{\tau<\eta} - (1-\tau)\right)\right|^2 d\tau\, d\eta\right] \\
&\qquad + 2\mathbb{E}\left[\int_0^1 \int_0^1 \sum_{\alpha,\beta} \left|\left(\left[\sigma_0^{-1}\right] D_{\tau,\alpha}\sigma_0\left[\sigma_0^{-1}\right]\right)_\theta^{\theta'} \mathcal{A}_{\theta'} \widetilde{\Phi}_\theta^\beta(\eta)\right|^2 d\tau\, d\eta\right] \\
&\quad \leqslant 2\mathbb{E}\left[\left\|\sigma_0^{-1}\right\|^2 \|\mathcal{A}\|^2 \|\Omega\|^2\right] + 2c\mathbb{E}\left[\left\|\sigma_0^{-1}\right\|^4 \|\mathcal{A}\|^2 \|\Omega\|^4\right].
\end{aligned}
$$

The final result follows from (3.5.1), Lemma 3.5.1, which shows that $t\delta(\zeta_0^t) = \delta(\zeta_0^1)$, where the superscript t and 1 is used to indicate the dependence on t explicitly, and the convergence

$$
\lim_{t\to 0} t\mathbb{E}\left[\left\|\delta\left(\zeta^t - \zeta_0^t\right)\right\|\right] \leqslant \lim_{t\to 0} t\left\|\zeta^t - \zeta_0^t\right\|_{W_1^2} = 0.
$$

The last convergence is a consequence of a.e. convergence of the integrands and an L^p, $p > 1$, uniform boundedness in t of them. Namely, from the estimates of the expectations of stochastic integrals, we have

$$\begin{aligned} t^2\mathbb{E}\|\zeta^t\|^2 &\leqslant t^2\mathbb{E}\int_0^t \|\sigma^{-1}\mathcal{A}\widetilde{\Phi}(\eta)\|^2 \\ &\leqslant t^2\|\mathcal{A}\|^2\left(\mathbb{E}\|\sigma^{-1}\|^4\right)^{1/2}\left(\mathbb{E}\left[\int_0^t \|\tilde{\Phi}(\eta)\|^2\right]^2\right)^{1/2} \\ &\leqslant C_4(\Omega)^{1/2}\|\mathcal{A}\|^2, \end{aligned}$$

where we have dropped the indices for the sake of simplicity.

Concerning the first order term, we have

$$\begin{aligned} t^2\mathbb{E}\|D\zeta^t\|^2 \leqslant{}& 2t^2\mathbb{E}\left(\int_0^t\int_0^t \|\sigma^{-1}\mathcal{A}D_\tau\widetilde{\Phi}(\eta)\|^2\,d\eta\right)^{1/2} \\ &+ 2t^2\mathbb{E}\left(\int_0^t\int_0^t \|\sigma^{-1}\|^2\|D_\tau\sigma\|^2\|\sigma^{-1}\|^2\|\mathcal{A}\|^2\|\widetilde{\Phi}\|^2\,d\eta\right)^{1/2}. \end{aligned}$$

This term is much more messy but can be treated in an analogous way, as long as we can rely on the Lemma 3.3.1. Indeed, by the expression for the derivation of stochastic integrals, the matrices $D_\tau\sigma$ has the same dependence on t as σ itself, which makes the estimation of all terms very much alike. □

3.6. Transversal analyticity

Let $(\Omega)_r$ be the endomorphism of $\Lambda^2(\mathbb{R}^d)$ defied by the curvature tensor Ω at $r \in O(M)$, and set

$$c_\Omega = \sup_{r\in O(M)} \left\|[(\Omega)_r]^{-1}\right\| \leqslant \infty.$$

Suppose that $c_\Omega < \infty$, i.e. $(\Omega)_r$ is invertible at every $r \in O(M)$. Let $\{L_\alpha\}$ be the system of the canonical horizontal vector fields on $O(M)$. Then $[L_\alpha, L_\beta]$ is vertical and $\Omega_{\alpha,\beta;\theta} = -\ddot{\Theta}_\theta([L_\alpha, L_\beta])$, $\ddot{\Theta}$ being the second component of the canonical parallelism Θ on $O(M)$. Thus the invertibility of $(\Omega)_r$ implies that $\{L_\alpha\}$ satisfies the Hörmander condition, and hence there exists the heat kernel corresponding to $\Delta_{O(M)}/2$, where $\Delta_{O(M)} = \sum_\alpha (L_\alpha)^2$ is the horizontal Laplacian on $O(M)$.

THEOREM 3.6.1. – *Let $\pi_t(r_0, r)$ be the heat kernel associated with $\Delta_{O(M)}/2$, the half of the holizontal Laplacian on $O(M)$. Then, for any $r_0 \in O(M)$ and $\mathcal{A} \in \mathfrak{so}(d)$, it holds that*

$$\limsup_{t\to 0} \frac{t}{\|\mathcal{A}\|} \int_{O(M)} |\partial_{\mathcal{A}} \pi_t(r_0, r)| v_{O(M)}(dr) \leqslant c_d c_\Omega, \tag{3.6.1}$$

where $\partial_{\mathcal{A}}$ stands for the vertical differentiation in the direction $\mathcal{A}$, $v_{O(M)}$ is the volume element of $O(M)$, and c_d is a universal constant depending only on the dimension d of M. Furthermore, the function $O(d) \ni g \mapsto \pi_t(r_0, gr)$ possesses a holomorphic extension to the open set

$$O_t = \{\gamma \in \mathrm{SG}L(\mathbb{C}^d)\colon\ \gamma = g \exp h,\ g \in \mathrm{SO}(d),\ \|h\| < t/(ec_d c_\Omega)\}.$$

Proof. – Applying an integration by parts on the Wiener space and Theorem 3.5.1, we obtain (3.6.1). To show that the second assertion follows from (3.6.1), see [6]. □

4. Asymptotic estimate of ground state

Following the method of [6], the transversal analyticity obtained in Theorem 3.6.1, implies asymptotic bounds for the ground states of fiber bundles associated with representations of the orthogonal groups with large dominant weights.

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